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ON THE OPTIMALITY OF A THEOREM OF ELTON ON ℓ_1^n SUBSYSTEMS

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ABSTRACT

We present examples which show that a substantial strengthening of the hypothesis in the almost isometric part of a theorem of Elton on ℓ_1^n subsystems does not lead to a substantially stronger conclusion.

A well-known theorem of Elton [1, Theorem 1] on the existence of ℓ_1^n subsystems is in two parts. The second part, which is 'almost isometric' in character, may be formulated as follows.

THEOREM E: *Let $\alpha \in (0, 1/2)$ and let $\beta \in (0,1)$. There exists $\delta < 1$ (depending only on α and β) such that if $(e_i)_{i=1}^n$ are vectors in the unit ball of a real Banach space X such that*

$$\text{(1)} \qquad \operatorname*{average}_{\pm} \left\| \sum_{i=1}^{n} \pm e_i \right\| \geq \delta n$$

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(the average taken over all choices of $\pm$*) then there exists* $A \subseteq \{1, 2, \ldots, n\}$*, with* $|A| \geq (1/2 - \alpha)n$*, such that*

$$\left\| \sum_{i \in A} a_i e_i \right\| \geq \beta \sum_{i \in A} |a_i|$$

for all scalars $(a_i)_{i \in A}$.

We are not concerned here with the first part of [1, Theorem 1], which is 'isomorphic' in character: the reader is referred to [1] for this result (and to [3] for the case of complex scalars); the reader is referred to [4] and [5] for further isomorphic results related to Theorem E.

An example due to Szarek [1, p. 121] shows that it is not possible to choose A in Theorem E to satisfy $|A| \geq (1/2 + \alpha)n$. The theorem below answers a question raised in [1, p. 114] by showing that it is not possible to choose A in Theorem E to satisfy $|A| \geq (1/2 + \alpha)n$ even if the hypothesis (1) is replaced by the stronger hypothesis

$$\min_{\pm} \left\| \sum_{i=1}^{n} \pm e_i \right\| \geq \delta n.$$

In fact, given $\beta > 0$, our example can be constructed (see (ii) of the Theorem) to satisfy

$$\min_{\pm} \left\| \sum_{i=1}^{n} \pm e_i \right\| \geq n - \beta.$$

Recall that a sequence $(y_i)_{i=1}^n$ in a Banach space X is **suppression 1-unconditional** if, whenever $A \subseteq B \subseteq \{1, 2, \ldots, n\}$, then $\| \sum_A a_i y_i \| \leq \| \sum_B a_i y_i \|$ for all scalars (a_i). In the following, $(e_i)_{i=1}^n$ denotes the standard basis of $\mathbb{R}^n$ and $\| \cdot \|_1$ denotes the ℓ_1^n norm. For a vector $x = \sum_{i=1}^n x_i e_i$, $\operatorname{supp} x$ denotes the set $\{1 \leq i \leq n \colon x_i \neq 0\}$.

THEOREM: *Let* $\alpha \in (0, 1/2)$ *and let* $\beta \in (0, 1)$*. For all sufficiently large* n *there exists a norm* $\| \cdot \|$ *on* $\mathbb{R}^n$ *with the following properties:*

(i) $(e_i)_{i=1}^n$ *is a suppression 1-unconditional normalized basis of* $(\mathbb{R}^n, \| \cdot \|)$.

(ii)
$$\left\| \sum_{i=1}^{n} \pm e_i \right\| \geq n - \beta$$
for all choices of signs.

(iii) *For every* $A \subseteq \{1, 2, \ldots, n\}$*, with* $|A| = 1 + \lceil (1/2 + \alpha)n \rceil$*, there exists a nonzero vector* x*, with* $\operatorname{supp} x \subseteq A$*, such that*

$$\|x\|_1 \geq (1 + \eta(\alpha, \beta)) \|x\|, \tag{2}$$

where

$$\eta(\alpha,\beta) = \frac{\alpha\beta}{(3\alpha+1)\lfloor(2\ln 2)/\alpha^2\rfloor - \alpha\beta}.$$

Remark: Note that $\eta(\alpha,\beta) \geq c\alpha^3\beta$, where c is an absolute constant. This *linear* dependence of η on β is optimal since (ii), $\|e_i\| \leq 1$, and the triangle inequality imply

$$\|x\| \geq (1-\beta)\|x\|_1 \quad (x \in \mathbb{R}^n).$$

The following probabilistic lemma will be used to construct $\|\cdot\|$. (Here $A\triangle B$ denotes the **symmetric difference** of A and B.)

LEMMA: *Let $\alpha \in (0,1)$. For all sufficiently large n there exist n sets $S_i \subseteq \{1,2,\dots,n\}$ $(1 \leq i \leq n)$ satisfying the following: for every $S \subseteq \{1,2,\dots,n\}$, we have*

$$|\{1 \leq i \leq n\colon \min(|S\triangle S_i|, |(I\setminus S)\triangle S_i|) \leq (1/2-\alpha/2)n\}| \leq \frac{2\ln 2}{\alpha^2}.$$

Proof: First we recall a well-known estimate (see, e.g., [2] for a more general inequality). Let $(\varepsilon_m)_{m=1}^{\infty}$ be a sequence of independent Bernoulli random variables (defined on a probability space $(\Omega,\mathbb{P})$) taking the values 1 and -1 with probability 1/2. Then, for $\alpha > 0$ and $n \geq 1$, we have

$$\mathbb{P}\left(\sum_{m=1}^{n} \varepsilon_m \geq \alpha n\right) \leq \exp(-n\alpha^2/2). \tag{3}$$

Set $k = \lfloor(2\ln 2)/\alpha^2\rfloor + 1$. We shall choose the sets S_i *independently* with the uniform distribution. Fix $S \subseteq \{1,2,\dots,n\}$. Then, for each $1 \leq i \leq n$, we have

$$\mathbb{P}(|S\triangle S_i(\omega)| \leq (1/2-\alpha/2)n) \leq \exp(-n\alpha^2/2).$$

Indeed, this is precisely equivalent to (3) if we identify subsets of $\{1,2,\dots,n\}$ with sequences of 1's and -1's in the obvious way. Hence

$$\mathbb{P}(\min(|S\triangle S_i(\omega)|, |(I\setminus S)\triangle S_i(\omega)|) \leq (1/2-\alpha/2)n) \leq 2\exp(-n\alpha^2/2).$$

Now fix $1 \leq j_1 < j_2 < \dots < j_k \leq n$. By independence, we have

$$\begin{aligned}\mathbb{P}(\min(|S\triangle S_i(\omega)|, |(I\setminus S)\triangle S_i(\omega)|) \leq (1/2-\alpha/2)n \text{ for all } i \in \{j_1,\dots,j_k\})\\ \leq 2^k\exp(-kn\alpha^2/2).\end{aligned}$$

So the probability that *there exists* $S \subseteq \{1,2,\dots,n\}$ and that there exist indices $1 \leq j_1 < j_2 < \dots < j_k \leq n$ for which

$$\min(|S\triangle S_i(\omega)|, |(I\setminus S)\triangle S_i(\omega)|) \leq (1/2-\alpha/2)n$$

for all $i \in \{j_1, \dots, j_k\}$ is at most

$$2^n \binom{n}{k} 2^k \exp(-kn\alpha^2/2) = \binom{n}{k} 2^k \exp(-n(k\alpha^2/2 - \ln 2)).$$

Since $k\alpha^2/2 - \ln 2 > 0$, this probability is less than 1 for all sufficiently large n. So, for all sufficiently large n, there exists $\omega \in \Omega$ such that $S_i(\omega)$ $(1 \le i \le n)$ satisfy the conclusion. ∎

Now we start on the proof of the Theorem. Let $I = \{1, 2, \dots, n\}$ and let $(S_i)_{i=1}^n$ satisfy the conclusion of the Lemma for $\alpha \in (0, 1/2)$. Let $k(\alpha) = \lfloor (2\ln 2)/\alpha^2 \rfloor$ and let $\gamma = \beta/k(\alpha)$. Note that $\gamma \in (0,1)$.

For $1 \le i \le n$, we say that a set $S \subseteq I$ is i-**large** if either $|S \triangle S_i| \le (1/2 - \alpha/2)n$ or $|(I \setminus S) \triangle S_i| \le (1/2 - \alpha/2)n$. Note that, for each $1 \le i \le n$, the collection of all i-large sets is closed under complementation.

Let $y = (y_i)_{i \in I}$ be a vector whose coordinates belong to the interval $[-1, 1]$. We set $P(y) = \{i \in I \colon y_i > 1 - \gamma\}$ and $N(y) = \{i \in I \colon y_i < -1 + \gamma\}$. For $S \subseteq I$, we say that y is S-**admissible** and that y **is obtained from** S if the following conditions hold:

(a) $|y_i| \le 1 - \gamma$ whenever S is i-large.

(b) $P(y) \subseteq S$ and $N(y) \subseteq I \setminus S$.

Note that if y is S-admissible then $-y$ is $(I \setminus S)$-admissible. This follows from the fact that the collection of i-large sets is closed under complementation.

A vector y is said to be **admissible** if y is S-admissible for some $S \subseteq I$. Let F denote the collection of all admissible vectors. Then F is symmetric, i.e., if $y \in F$ then $-y \in F$.

Now we can define the norm $\|\cdot\|$:

$$\left\| \sum_{i \in I} x_i e_i \right\| = \max_{y \in F} \sum_{i \in I} x_i y_i. \tag{4}$$

The symmetry of F guarantees that (4) defines a norm. The fact that this norm is suppression 1-unconditional is an immediate consequence of the following easily checked property of F: if $y \in F$ and z is obtained from y by replacing some of the coordinates of y by zeros, then $z \in F$. It is also easy to check that $\|e_i\| = 1$ for all $1 \le i \le n$.

Proof of (ii): Let $\eta = (\eta_i)_{i=1}^n$ be a choice of signs. Define $y = (y_i)$ thus:

$$y_i = \begin{cases} \eta_i & \text{if } P(\eta) \text{ is } \textit{not } i\text{-large}, \\ (1-\gamma)\eta_i & \text{if } P(\eta) \text{ is } i\text{-large}. \end{cases}$$

Clearly, y is $P(\eta)$-admissible, so $y \in F$. By the Lemma, $P(\eta)$ is i-large for at most $k(\alpha)$ indices i. Thus

$$\left\|\sum_{i=1}^{n} \eta_i e_i\right\| \geq \sum_{i=1}^{n} \eta_i y_i \geq \sum_{i=1}^{n} \eta_i^2 - k(\alpha)\gamma = n - \beta. \quad \blacksquare$$

Proof of (iii): Suppose $A \subset I$ with $|A| = 1 + \lceil (1/2+\alpha)n \rceil$. Choose $i_0 \in A$ and set $\tilde{A} = A \setminus \{i_0\}$ (so that $|\tilde{A}| = \lceil (1/2+\alpha)n \rceil$). We define a vector x, with $\operatorname{supp} x = A$, thus:

$$x_i = \begin{cases} |\tilde{A}| - (1/2+\alpha/2)n & \text{for } i = i_0, \\ 1 & \text{for } i \in \tilde{A} \cap S_{i_0}, \\ -1 & \text{for } i \in \tilde{A} \cap (I \setminus S_{i_0}), \\ 0 & \text{otherwise.} \end{cases}$$

Now let us show that $\|x\|$ satisfies (2). Let y be an admissible vector that is obtained from $S \subseteq I$. Suppose that

$$|\tilde{A} \cap S_{i_0} \cap P(y)| + |\tilde{A} \cap (I \setminus S_{i_0}) \cap N(y)| > (1/2+\alpha/2)n. \tag{5}$$

Since $P(y) \subseteq S$ and $N(y) \subseteq I \setminus S$, we have

$$|S_{i_0} \cap S| + |(I \setminus S_{i_0}) \cap (I \setminus S)| > (1/2+\alpha/2)n.$$

Thus,

$$|S_{i_0} \triangle S| < (1/2-\alpha/2)n.$$

So S is i_0-large. Thus $|y_{i_0}| \leq 1-\gamma$. Hence

$$\begin{aligned} \sum_{i \in I} x_i y_i &= x_{i_0} y_{i_0} + \sum_{i \in \tilde{A}} x_i y_i \\ &\leq (1-\gamma)(|\tilde{A}| - (1/2+\alpha/2)n) + |\tilde{A}|. \end{aligned} \tag{6}$$

Note that if $i \in \tilde{A} \setminus ((\tilde{A} \cap S_{i_0} \cap P(y)) \cup (\tilde{A} \cap (I \setminus S_{i_0}) \cap N(y)))$, then $x_i y_i \leq 1-\gamma$. It follows that if (5) does not hold, then

$$\begin{aligned} \sum_{i \in I} x_i y_i &\leq |x_{i_0}| + |\tilde{A}| - \gamma(|\tilde{A}| - (1/2+\alpha/2)n) \\ &= (|\tilde{A}| - (1/2+\alpha/2)n) + |\tilde{A}| - \gamma(|\tilde{A}| - (1/2+\alpha/2)n) \\ &= (1-\gamma)(|\tilde{A}| - (1/2+\alpha/2)n) + |\tilde{A}|. \end{aligned} \tag{7}$$

It follows from (6) and (7) that

$$\|x\| = \sup_{y \in F} \sum_{i=1}^{n} x_i y_i \leq (1-\gamma)(|\tilde{A}| - (1/2+\alpha/2)n) + |\tilde{A}|. \tag{8}$$

But

$$\|x\|_1 = |x_{i_0}| + |\tilde{A}|$$
$$= (|\tilde{A}| - (1/2 + \alpha/2)n) + |\tilde{A}|$$
$$\geq \|x\| + \gamma(|\tilde{A}| - (1/2 + \alpha/2)n)$$

(by (8))

$$\geq \left(1 + \frac{\gamma(|\tilde{A}| - (1/2 + \alpha/2)n)}{(1-\gamma)(|\tilde{A}| - (1/2 + \alpha/2)n) + |\tilde{A}|}\right) \|x\|$$

(by (8) again)

$$\geq \left(1 + \gamma\left\{(1-\gamma) + \frac{1+2\alpha}{\alpha}\right\}^{-1}\right) \|x\|$$

since $|\tilde{A}| \geq (1/2 + \alpha)n$. Substituting $k(\alpha) = \lfloor (2\ln 2)/\alpha^2 \rfloor$ and $\gamma = \beta/k(\alpha)$ into the above inequality yields (2). ∎

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